Chapter 4: Basic Constraint Reasoning
(SEND+MORE=MONEY)

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ECLiPSe ELearning

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What we want to introduce

- Finite Domain Solver in ECLiPSe
- Models and Programs
- Constraint Propagation and Search
- Basic constraints: linear arithmetic, alldifferent, disequality
- Built-in search: Labeling
- Visualizers for variables, constraints and search
A Crypt-Arithmetic Puzzle

We begin with the definition of the SEND+MORE=MONEY puzzle. It is often shown in the form of a hand-written addition:

\[
\begin{array}{c}
S & E & N & D \\
+ & M & O & R & E \\
\hline
M & O & N & E & Y
\end{array}
\]

Rules

- Each character stands for a digit from 0 to 9.
- Numbers are built from digits in the usual, positional notation.
- Repeated occurrence of the same character denote the same digit.
- Different characters denote different digits.
- Numbers do not start with a zero.
- The equation must hold.
Each character is a variable, which ranges over the values 0 to 9.

An *alldifferent* constraint between all variables, which states that two different variables must have different values. This is a very common constraint, which we will encounter in many other problems later on.

Two *disequality constraints* (variable $X$ must be different from value $V$) stating that the variables at the beginning of a number can not take the value 0.

An arithmetic *equality constraint* linking all variables with the proper coefficients and stating that the equation must hold.

```prolog
:- module(sendmory).
:- export(sendmory/1).
:- lib(ic).
sendmory(L):-
    L = [S,E,N,D,M,O,R,Y], % Variables
    L :: 0..9,
    alldifferent(L), % Constraints
    S #\= 0, M #\= 0,
    1000*S + 100*E + 10*N + D +
    1000*M + 100*O + 10*R + E #=
    10000*M + 1000*O + 100*N + 10*E + Y,
    labeling(L). % Search
```
Choice of Model

- This is *one* model, not *the* model of the problem
- Many possible alternatives
- Choice often depends on your constraint system
  - Constraints available
  - Reasoning attached to constraints
- Not always clear which is the *best* model
- Often: Not clear what is the *problem*

Running the program

- To run the program, we have to enter the query
  - `sendmory:sendmory(L)`.
- Result
  - `L = [9, 5, 6, 7, 1, 0, 8, 2]`
  - `yes (0.00s cpu, solution 1, maybe more)`
But how did the program come up with this solution?

\[
L = [S, E, N, D, M, O, R, Y], \\
L :: 0..9, \\
[S, E, N, D, M, O, R, Y] \in \{0..9\}
\]
Problem  
    Program  
    Constraint Setup  
    Search  
    Lessons Learned

Domain Definition  
    Alldifferent Constraint  
    Disequality Constraints  
    Equality Constraint

Domain Visualization

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Rows = Variables  
Columns = Values

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Alldifferent Constraint

\[
\text{alldifferent}(L),
\]

- Built-in of \textit{ic} library
- No initial propagation possible
- \textit{Suspends}, waits until variables are changed
- When variable is fixed, remove value from domain of other variables
- \textit{Forward checking}
Alldifferent Visualization

Uses the same representation as the domain visualizer

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Disequality Constraints

\[ S \#\neq 0, \ M\#\neq 0, \]

Remove value from domain

\[ S \in \{1..9\}, \ M \in \{1..9\} \]

Constraints solved, can be removed
Domains after Disequality

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Equality Constraint

- Normalization of linear terms
  - Single occurrence of variable
  - Positive coefficients
- Propagation
Normalization

\[
\begin{align*}
1000S &+ 100E + 10N + D \\
+1000M &+ 100O + 10R + E \\
10000M &+ 1000O + 100N + 10E + Y
\end{align*}
\]

is transformed into

\[
\begin{align*}
1000S &+ 91E + 10R + D \\
+ &10E \\
9000M &+ 900O + 90N + Y
\end{align*}
\]

Simplified Equation

\[
1000 * S + 91 * E + 10 * R + D = 9000 * M + 900 * O + 90 * N + Y
\]
Consider lower bound for $S$

\[ 1000 \times S^{1..9} + 91 \times E^{0..9} + 10 \times R^{0..9} + D^{0..9} = 9000 \times M^{1..9} + 900 \times O^{0..9} + 90 \times N^{0..9} + Y^{0..9} \]

- Lower bound of equation is 9000
- Rest of Lhs (left hand side) $(91 \times E^{0..9} + 10 \times R^{0..9} + D^{0..9})$ is almost 918
- $S$ must be greater or equal to $\frac{9000 - 918}{1000} = 8.082$
  - otherwise lower bound of equation not reached by Lhs
- $S$ is integer, therefore $S \geq \lceil \frac{9000 - 918}{1000} \rceil = 9$
- $S$ has upper bound of 9, so $S = 9$
Consider upper bound of $M$

\[
\frac{1000 \times S^{1.9} + 91 \times E^{0.9} + 10 \times R^{0.9} + D^{0.9}}{9000..9918} = \frac{9000 \times M^{1.9} + 900 \times O^{0.9} + 90 \times N^{0.9} + Y^{0.9}}{9000..9918}
\]

- Upper bound of equation is 9918
- Rest of rhs (right hand side) $900 \times O^{0.9} + 90 \times N^{0.9} + Y^{0.9}$ is at least 0
- $M$ must be smaller or equal to $\frac{9918 - 0}{9000} = 1.102$
- $M$ must be integer, therefore $M \leq \lfloor \frac{9918 - 0}{9000} \rfloor = 1$
- $M$ has lower bound of 1, so $M = 1$

---

Consider upper bound of $O$

\[
\frac{1000 \times S^{1.9} + 91 \times E^{0.9} + 10 \times R^{0.9} + D^{0.9}}{9000..9918} = \frac{9000 \times M^{1.9} + 900 \times O^{0.9} + 90 \times N^{0.9} + Y^{0.9}}{9000..9918}
\]

- Upper bound of equation is 9918
- Rest of rhs (right hand side) $9000 \times 1 + 90 \times N^{0.9} + Y^{0.9}$ is at least 9000
- $O$ must be smaller or equal to $\frac{9918 - 9000}{900} = 1.02$
- $O$ must be integer, therefore $O \leq \lfloor \frac{9918 - 9000}{900} \rfloor = 1$
- $O$ has lower bound of 0, so $O \in \{0..1\}$
### Propagation of equality: Result

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### Propagation of alldifferent

\[ O = 0, [E, R, D, N, Y] \in \{2..8\}\]
Waking the equality constraint

- Triggered by assignment of variables
- or update of lower or upper bound

Removal of constants

\[
1000 \times 9 + 91 \times E^{2..8} + 10 \times R^{2..8} + D^{2..8} = \\
9000 \times 1 + 900 \times 0 + 90 \times N^{2..8} + Y^{2..8}
\]

\[
1000 \times 9 + 91 \times E^{2..8} + 10 \times R^{2..8} + D^{2..8} = \\
9000 \times 1 + 900 \times 0 + 90 \times N^{2..8} + Y^{2..8}
\]

\[
91 \times E^{2..8} + 10 \times R^{2..8} + D^{2..8} = 90 \times N^{2..8} + Y^{2..8}
\]
**Propagation of equality (Iteration 1)**

\[
\begin{align*}
91 \cdot E^{2.8} + 10 \cdot R^{2.8} + D^{2.8} &= 90 \cdot N^{2.8} + Y^{2.8} \\
204.816 &= 182.728 \\
91 \cdot E^{2.8} + 10 \cdot R^{2.8} + D^{2.8} &= 90 \cdot N^{2.8} + Y^{2.8} \\
204.728
\end{align*}
\]

\[
N \geq 3 = \left\lceil \frac{204 - 8}{90} \right\rceil, \quad E \leq 7 = \left\lfloor \frac{728 - 22}{91} \right\rfloor
\]

**Propagation of equality (Iteration 2)**

\[
\begin{align*}
91 \cdot E^{2.7} + 10 \cdot R^{2.8} + D^{2.8} &= 90 \cdot N^{3.8} + Y^{2.8} \\
204.725 &= 272.728 \\
91 \cdot E^{2.7} + 10 \cdot R^{2.8} + D^{2.8} &= 90 \cdot N^{3.8} + Y^{2.8} \\
272.725
\end{align*}
\]

\[
E \geq 3 = \left\lceil \frac{272 - 88}{91} \right\rceil
\]
Propagation of equality (Iteration 3)

\[
91 \cdot E^{3..7} + 10 \cdot R^{2..8} + D^{2..8} = 90 \cdot N^{3..8} + Y^{2..8}
\]

\[
\frac{91 \cdot E^{3..7} + 10 \cdot R^{2..8} + D^{2..8}}{295.725} = \frac{90 \cdot N^{3..8} + Y^{2..8}}{272.728}
\]

\[
\frac{91 \cdot E^{3..7} + 10 \cdot R^{2..8} + D^{2..8}}{295.725} = \frac{90 \cdot N^{3..8} + Y^{2..8}}{272.728}
\]

\[
N \geq 4 = \left\lfloor \frac{295 - 8}{90} \right\rfloor
\]

Propagation of equality (Iteration 4)

\[
91 \cdot E^{3..7} + 10 \cdot R^{2..8} + D^{2..8} = 90 \cdot N^{4..8} + Y^{2..8}
\]

\[
\frac{91 \cdot E^{3..7} + 10 \cdot R^{2..8} + D^{2..8}}{295.725} = \frac{90 \cdot N^{4..8} + Y^{2..8}}{362.728}
\]

\[
\frac{91 \cdot E^{3..7} + 10 \cdot R^{2..8} + D^{2..8}}{362.725} = \frac{90 \cdot N^{4..8} + Y^{2..8}}{362.728}
\]

\[
E \geq 4 = \left\lfloor \frac{362 - 88}{91} \right\rfloor
\]
Propagation of equality (Iteration 5)

\[
91 \cdot E^{4..7} + 10 \cdot R^{2..8} + D^{2..8} = 90 \cdot N^{4..8} + Y^{2..8}
\]

\[
\frac{91 \cdot E^{4..7} + 10 \cdot R^{2..8} + D^{2..8}}{386..725} = \frac{90 \cdot N^{4..8} + Y^{2..8}}{362..728}
\]

\[
\frac{91 \cdot E^{4..7} + 10 \cdot R^{2..8} + D^{2..8}}{386..725} = 90 \cdot N^{5..8} + Y^{2..8}
\]

\[
N \geq 5 = \left\lceil \frac{386 - 8}{90} \right\rceil
\]

No further propagation at this point.
labeling built-in

```
labeling([S,E,N,D,M,O,R,Y])

- Try variable is order given
- Try values starting from smallest value in domain
- When failing, backtrack to last open choice
- Chronological Backtracking
- Depth First search
```
### Search Tree Step 1

Variable $S$ already fixed

### Step 2, Alternative $E = 4$

Variable $E \in \{4..7\}$, first value tested is 4
Assignment $E = 4$

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Propagation of $E = 4$, equality constraint

\[
91 \times 4 + 10 \times R^{2..8} + D^{2..8} = 90 \times N^{5..8} + Y^{2..8}
\]

\[
91 \times 4 + 10 \times R^{2..8} + D^{2..8} = 90 \times N^{5..8} + Y^{2..8}
\]

\[
91 \times 4 + 10 \times R^{2..8} + D^{2..8} = 90 \times N^{5..8} + Y^{2..8}
\]

$N = 5, Y = 2, R = 8, D = 8$
### Result of equality propagation

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### Propagation of alldifferent

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**Alldifferent fails!**
Step 2, Alternative $E = 5$

Return to last open choice, $E$, and test next value

Assignment $E = 5$
Propagation of alldifferent

\[
\begin{array}{cccccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
S & & & & & & & & & \\
E & & & & & & & & & \\
N & & & & & & & & & \\
D & & & & & & & & & \\
M & & & & & & & & & \\
O & & & & & & & & & \\
R & & & & & & & & & \\
Y & & & & & & & & & \\
\end{array}
\]

\[N \neq 5, N \geq 6\]

Propagation of equality

\[
91 \times 5 + 10 \times R^{2..8} + D^{2..8} = 90 \times N^{6..8} + Y^{2..8}
\]

\[
\underbrace{91 \times 5 + 10 \times R^{2..8} + D^{2..8}}_{477..543} = \underbrace{90 \times N^{6..8} + Y^{2..8}}_{542..728}
\]

\[
\underbrace{91 \times 5 + 10 \times R^{2..8} + D^{2..8}}_{542..543} = \underbrace{90 \times N^{6..8} + Y^{2..8}}_{542..543}
\]

\[N = 6, Y \in \{2, 3\}, R = 8, D \in \{7..8\}\]
### Result of equality propagation

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### Propagation of `alldifferent`

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\[ D = 7 \]
Propagation of equality

\[ 91 \times 5 + 10 \times 8 + 7 = 90 \times 6 + Y^{2..3} \]

\[ 91 \times 5 + 10 \times 8 + 7 = 90 \times 6 + Y^{2..3} \]

\[ 91 \times 5 + 10 \times 8 + 7 = 90 \times 6 + Y^{2..3} \]

\[ Y = 2 \]

Last propagation step

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Helmut Simonis  Basic Constraint Reasoning
Complete Search Tree

Solution

\[ \begin{array}{cccccc}
9 & 5 & 6 & 7 \\
+ & 1 & 0 & 8 & 5 \\
\hline
1 & 0 & 6 & 5 & 2 \\
\end{array} \]
Topics introduced

- Finite Domain Solver in ECLiPSe, `ic` library
- Models and Programs
- Constraint Propagation and Search
- Basic constraints: linear arithmetic, `alldifferent`, disequality
- Built-in search: `labeling`
- Visualizers for variables, constraints and search

Lessons Learned

- Constraint models are expressed by variables and constraints.
- Problems can have many different models, which can behave quite differently. Choosing the best model is an art.
- Constraints can take many different forms.
- Propagation deals with the interaction of variables and constraints.
- It removes some values that are inconsistent with a constraint from the domain of a variable.
- Constraints only communicate via shared variables.
Lessons Learned

- Propagation usually is not sufficient, search may be required to find a solution.
- Propagation is data driven, and can be quite complex even for small examples.
- The default search uses chronological depth-first backtracking, systematically exploring the complete search space.
- The search choices and propagation are interleaved, after every choice some more propagation may further reduce the problem.

Alternative 1

- Do we need the constraint “Numbers do not begin with a zero”?
- This is not given explicitly in the problem statement
- Remove disequality constraints from program
- Previous solution is still a solution
- Does it change propagation?
- Does it have more solutions?
Program without Disequality

Listing 1: Alternative 1

```prolog
:- module(alternative1).
:- export(sendmory/1).
:- lib(ic).

sendmory(L):-
    L = [S,E,N,D,M,O,R,Y],
    L :: 0..9,
    allDifferent(L),
    1000*S + 100*E + 10*N + D +
    1000*M + 100*O + 10*R + E #= 
    10000*M + 1000*O + 100*N + 10*E + Y,
    labeling(L).
```

After Setup without Disequality

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Setup Comparison

Alternative Models
Exercises

Search Tree: Many Solutions

Model without Disequality
Multiple Equations

Search Tree: Many Solutions
Note:

- Not just a different model, solving a different problem!
- Often we can choose which problem we want to solve
  - Which constraints to include
  - What to ignore
- In this case not acceptable

Alternative 2

- Large equality difficult to understand by humans
- Replace with multiple, simpler equations
- Linked by carry variables (0/1)
- Should produce same solutions
- Does it give same propagation?

\[
\begin{array}{c}
S & E & N & D \\
+ & M & O & R & E \\
+ & C_5 & C_4 & C_3 & C_2 \\
\hline
M & O & N & E & Y
\end{array}
\]
Carry Variables with Multiple Equations

:-module(alternative2), export(sendmory/1), lib(ic).
sendmory(L):~⇒ same as before
    L=[S,E,N,D,M,O,R,Y], L :: 0..9,
    [C2,C3,C4,C5] :: 0..1, ~⇒ new
    alldifferent(L),
    S \= 0, M \= 0,
    M = C5,
    S+M+C4 = 10*C5+O,
    E+O+C3 = 10*C4+N,
    N+R+C2 = 10*C3+E,
    D+E = 10*C2+Y,
    labeling(L).

With Carry Variables: After Setup

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Setup Comparison

**Alternative Models**

**Exercises**

**Model without Disequality**

**Multiple Equations**

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**Search Tree: First Solution**

![Search Tree Diagram]

---

**Helmut Simonis**

Basic Constraint Reasoning

65

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**Helmut Simonis**

Basic Constraint Reasoning

66
Comparison

Single Equation

Multiple Equations

Observations

- This is solving the original problem
- Search tree slightly bigger
- Caused here by missing interaction of equations
- And repeated variables
- But: Introducing auxiliary variables not always bad!
Henry Dudeney. 
Send+More=Money. 
*Strand Magazine*, Volume 68:pages 97 and 214, July 1924.

Henry Dudeney. 
*Amusements in Mathematics.* 
Project Gutenberg, 1917. 
http://www.gutenberg.org/etext/16713.

---

### Exercises

1. Does the reasoning for the equality constraints that we have presented remove all inconsistent values? Consider the constraint $Y=2*X$.

2. Why is it important to remove multiple occurrences of the same variable from an equality constraint? Give an example!

3. Solve the puzzle DONALD+GERALD=ROBERT. What is the state of the variables before the search, after the initial constraint propagation?

4. Solve the puzzle $Y*WORRY = DOOOOD$. What is different?

5. (extra credit) How would you design a program that finds new crypt-arithmetic puzzles? What makes a good puzzle?