Chapter 4: Basic Constraint Reasoning  
(SEND + MORE = MONEY)
This work is licensed under the Creative Commons Attribution-Noncommercial-Share Alike 3.0 Unported License. To view a copy of this license, visit http://creativecommons.org/licenses/by-nc-sa/3.0/ or send a letter to Creative Commons, 171 Second Street, Suite 300, San Francisco, California, 94105, USA.
Outline

1. Problem
2. Program
3. Constraint Setup
4. Search
5. Lessons Learned
What we want to introduce

- Finite Domain Solver in ECLiPSe
- Models and Programs
- Constraint Propagation and Search
- Basic constraints: linear arithmetic, alldifferent, disequality
- Built-in search: Labeling
- Visualizers for variables, constraints and search
We begin with the definition of the SEND+MORE=MONEY puzzle. It is often shown in the form of a hand-written addition:
Each character stands for a digit from 0 to 9.

Numbers are built from digits in the usual, positional notation.

Repeated occurrence of the same character denote the same digit.

Different characters denote different digits.

Numbers do not start with a zero.

The equation must hold.
Outline

1. Problem
2. Program
3. Constraint Setup
4. Search
5. Lessons Learned
Model

- Each character is a variable, which ranges over the values 0 to 9.

- An *alldifferent* constraint between all variables, which states that two different variables must have different values. This is a very common constraint, which we will encounter in many other problems later on.

- Two *disequality constraints* (variable $X$ must be different from value $V$) stating that the variables at the beginning of a number can not take the value 0.

- An arithmetic *equality constraint* linking all variables with the proper coefficients and stating that the equation must hold.
Program Sendmory

:- module(sendmory).

Define Module

:- export(sendmory/1).

:- lib(ic).

sendmory(L):-
L = [S,E,N,D,M,O,R,Y],
L :: 0..9,
alldifferent(L),
S #\= 0, M #\= 0,
1000*S + 100*E + 10*N + D +
1000*M + 100*O + 10*R + E #= 
10000*M + 1000*O + 100*N + 10*E + Y,
labeling(L).
\[-\textbf{module} (sendmory).\]
\[-\textbf{export} (sendmory/1).\]
\[-\textbf{lib} (ic).\]
\[
\text{sendmory} (L):= \\
\text{L} = [S,E,N,D,M,O,R,Y], \\
\text{L :: 0..9,} \\
\textit{alldifferent} (L), \\
S \#\not= 0, M \#\not= 0, \\
1000\times S + 100\times E + 10\times N + D + \\
1000\times M + 100\times O + 10\times R + E \not= \\
10000\times M + 1000\times O + 100\times N + 10\times E + Y, \\
\textit{labeling} (L).
\]
Program Sendmory

:- module(sendmory).
:- export(sendmory/1).
:- lib(ic).

Use ic library

sendmory(L):-
    L = [S,E,N,D,M,O,R,Y],
    L :: 0..9,
    alldifferent(L),
    S #\= 0, M #\= 0,
    1000*S + 100*E + 10*N + D +
    1000*M + 100*O + 10*R + E #=
    10000*M + 1000*O + 100*N + 10*E + Y,
    labeling(L).
:- module(sendmory).
:- export(sendmory/1).
:- lib(ic).

sendmory(L):= Predicate definition
    L = [S,E,N,D,M,O,R,Y],
    L :: 0..9,
    alldifferent(L),
    S \= 0, M \= 0,
    1000*S + 100*E + 10*N + D +
    1000*M + 100*O + 10*R + E #=
    10000*M + 1000*O + 100*N + 10*E + Y,
    labeling(L).
:- module(sendmory).
:- export(sendmory/1).
:- lib(ic).

sendmory(L):-
    L = [S,E,N,D,M,O,R,Y], \ Define list
    L :: 0..9,
    alldifferent(L),
    S #\= 0, M #\= 0,
    1000*S + 100*E + 10*N + D +
    1000*M + 100*O + 10*R + E #=
    10000*M + 1000*O + 100*N + 10*E + Y,
    labeling(L).

\ H Helmut Simonis

\ B Basic Constraint Reasoning
Program Sendmory

:- module(sendmory).
:- export(sendmory/1).
:- lib(ic).

sendmory(L):-
    L = [S,E,N,D,M,O,R,Y],
    L :: 0..9,      \Define integer domain 0..9
    alldifferent(L),
    S \#\= 0, M \#\= 0,
    1000*S + 100*E + 10*N + D +
    1000*M + 100*O + 10*R + E #=
    10000*M + 1000*O + 100*N + 10*E + Y,
    labeling(L).

Helmut Simonis
Program Sendmory

:- module(sendmory).
:- export(sendmory/1).
:- lib(ic).

sendmory(L):-
    L = [S,E,N,D,M,O,R,Y],
    L :: 0..9,
    alldifferent(L),  % Digits must be different
    S #\= 0, M #\= 0,
    1000*S + 100*E + 10*N + D +
    1000*M + 100*O + 10*R + E #= 10000*M + 1000*O + 100*N + 10*E + Y,
    labeling(L).
Program Sendmory

:- module(sendmory).
:- export(sendmory/1).
:- lib(ic).

sendmory(L):-
    L = [S,E,N,D,M,O,R,Y],
    L :: 0..9,
    alldifferent(L),
    S #\= 0, M #\= 0, ☐ Numbers don’t start with 0
    1000*S + 100*E + 10*N + D +
    1000*M + 100*O + 10*R + E #=
    10000*M + 1000*O + 100*N + 10*E + Y,
    labeling(L).
Program Sendmory

:- module(sendmory).
:- export(sendmory/1).
:- lib(ic).

sendmory(L):-
  L = [S,E,N,D,M,O,R,Y],
  L :: 0..9,
  alldifferent(L),
  S #\= 0, M #\= 0,
  1000*S + 100*E + 10*N + D +
  1000*M + 100*O + 10*R + E #=*
  10000*M + 1000*O + 100*N + 10*E + Y,
  labeling(L).
:- module(sendmory).
:- export(sendmory/1).
:- lib(ic).

sendmory(L):-
L = [S,E,N,D,M,O,R,Y],
L :: 0..9,
alldifferent(L),
S #\= 0, M #\= 0,
1000*S + 100*E + 10*N + D + 
1000*M + 100*O + 10*R + E #= 
10000*M + 1000*O + 100*N + 10*E + Y, 
labeling(L).
⇒ built-in search routine
Program Sendmory

:- module(sendmory).
:- export(sendmory/1).
:- lib(ic).

sendmory(L):=

definition must match
L = [S,E,N,D,M,O,R,Y],
L :: 0..9,
alldifferent(L),
S #\= 0, M #\= 0,
1000*S + 100*E + 10*N + D +
1000*M + 100*O + 10*R + E #=
10000*M + 1000*O + 100*N + 10*E + Y,
labeling(L).
:- module(sendmory).
:- export(sendmory/1).
:- lib(ic).

sendmory(L):= \[
\text{for predicate definition}
\]
L = [S,E,N,D,M,O,R,Y],
L :: 0..9,
\[\text{alldifferent}\ (L)\],
S \#\!= 0, M \#\!= 0,
1000*S + 100*E + 10*N + D +
1000*M + 100*O + 10*R + E \=#
10000*M + 1000*O + 100*N + 10*E + Y,
\[\text{labeling}\ (L)\].
Program Sendmory

:- module(sendmory).
:- export(sendmory/1).
:- lib(ic).

sendmory(L):-
    L = [S,E,N,D,M,O,R,Y],
    L :: 0..9,
    alldifferent(L),
    S #\= 0, M #\= 0,
    1000*S + 100*E + 10*N + D +
    1000*M + 100*O + 10*R + E #=
    10000*M + 1000*O + 100*N + 10*E + Y,
    labeling(L).
Program Sendmory

:- module(sendmory).
:- export(sendmory/1).
:- lib(ic).

sendmory(L):-
    L = [S,E,N,D,M,O,R,Y], \ Confusing name!
    L :: 0..9,
    alldifferent(L),
    S \= 0, M \= 0,
    1000*S + 100*E + 10*N + D +
    1000*M + 100*O + 10*R + E ≠
    10000*M + 1000*O + 100*N + 10*E + Y,
    labeling(L).

Helmut Simonis  Basic Constraint Reasoning
Program Sendmory

:- module(sendmory).
:- export(sendmory/1).
:- lib(ic).

sendmory(L):-
    L = [S,E,N,D,M,O,R,Y],
    L :: 0..9,
    alldifferent(L),
    S #\= 0, M #\= 0,
    1000*S + 100*E + 10*N + D +
    1000*M + 100*O + 10*R + E #= 
    10000*M + 1000*O + 100*N + 10*E + Y, 
    labeling(L).
General Program Structure

:- module(sendmory).
:- export(sendmory/1).
:- lib(ic).
sendmory(L):-

    L = [S,E,N,D,M,O,R,Y], Variables
    L :: 0..9,
    alldifferent(L),
    S \= 0, M \= 0,
    1000*S + 100*E + 10*N + D +
    1000*M + 100*O + 10*R + E #=
    10000*M + 1000*O + 100*N + 10*E + Y,
    labeling(L).
General Program Structure

:- module(sendmory).
:- export(sendmory/1).
:- lib(ic).

sendmory(L):-
    L = [S,E,N,D,M,O,R,Y],
    L :: 0..9,
    alldifferent(L),  \ Constraints
    S #\= 0, M #\= 0,
    1000*S + 100*E + 10*N + D +
    1000*M + 100*O + 10*R + E #=
    10000*M + 1000*O + 100*N + 10*E + Y,
    labeling(L).
:- module(sendmory).
:- export(sendmory/1).
:- lib(ic).

sendmory(L):-
    L = [S,E,N,D,M,O,R,Y],
    L :: 0..9,
    alldifferent(L),
    S #\= 0, M #\= 0,
    1000*S + 100*E + 10*N + D +
    1000*M + 100*O + 10*R + E #= 
    10000*M + 1000*O + 100*N + 10*E + Y,
    labeling(L).

³ Search
Choice of Model

- This is *one* model, not *the* model of the problem
- Many possible alternatives
- Choice often depends on your constraint system
  - Constraints available
  - Reasoning attached to constraints
- Not always clear which is the *best* model
- Often: Not clear what is the *problem*

Alternative 1  Alternative 2
Running the program

To run the program, we have to enter the query

```
sendmory:sendmory(L).
```

Result

- \( L = [9, 5, 6, 7, 1, 0, 8, 2] \)
- yes (0.00s cpu, solution 1, maybe more)
But how did the program come up with this solution?
Outline

1. Problem
2. Program
3. Constraint Setup
   - Domain Definition
   - Alldifferent Constraint
   - Disequality Constraints
   - Equality Constraint
4. Search
5. Lessons Learned
Domain Definition

\[ L = [S, E, N, D, M, O, R, Y], \]
\[ L :: 0..9, \]

\[ [S, E, N, D, M, O, R, Y] \in \{0..9\} \]
## Domain Visualization

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>S</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>E</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>N</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>D</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>M</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>O</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>R</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Y</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Domain Visualization

```
<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>S</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>E</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>N</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>D</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>M</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>O</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>R</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Y</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
```

Rows = Variables
Domain Visualization

Columns = Values

<table>
<thead>
<tr>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>S</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>E</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>N</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>D</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>M</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>O</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>R</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Y</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Domain Visualization

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>S</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>E</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>N</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>D</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>M</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Cells= State</td>
<td></td>
</tr>
<tr>
<td>O</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>R</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Y</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Cells = State
Alldifferent Constraint

\texttt{alldifferent(L)},

- Built-in of \texttt{ic} library
- No initial propagation possible
- \textit{Suspends}, waits until variables are changed
- When variable is fixed, remove value from domain of other variables
- \textit{Forward checking}
Uses the same representation as the domain visualizer

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>S</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>E</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>N</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>D</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>M</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>O</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>R</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Y</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Disequality Constraints

\[ S \neq 0, \; M \neq 0, \]

Remove value from domain

\[ S \in \{1..9\}, \; M \in \{1..9\} \]

Constraints solved, can be removed
## Domains after Disequality

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>S</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>E</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>N</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>D</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>M</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>O</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>R</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Y</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Equality Constraint

- Normalization of linear terms
  - Single occurrence of variable
  - Positive coefficients
- Propagation
Normalization

\[
\begin{align*}
1000S + 100E + 10N + D \\
+ 1000M + 100O + 10R + E \\
\hline
10000M + 1000O + 100N + 10E + Y
\end{align*}
\]
Normalization

\[
\begin{align*}
1000\times S &+ 100\times E &+ 10\times N &+ D \\
+ 1000\times M & & & \\
\hline
10000\times M &+ 1000\times O &+ 100\times N &+ 10\times E &+ Y
\end{align*}
\]
Normalization

\[
\begin{align*}
1000*S &+ 100*E &+ 10*N &+ D \\
+ 100*O &+ 10*R &+ E \\
9000*M &+ 1000*O &+ 100*N &+ 10*E &+ Y
\end{align*}
\]
Normalization

\[ 1000S + 100E + 10N + D \]
\[ + 100O + 10R + E \]
\[ = 9000M + 1000O + 100N + 10E + Y \]
Normalization

\[
\begin{array}{cccccc}
1000*S+ & 100*E+ & 10*N+ & D & \\
& & + & 10*R+ & E & \\
9000*M+ & 900*O+ & 100*N+ & 10*E+ & Y \\
\end{array}
\]
Normalization

\[
\begin{align*}
1000*S &+ 100*E &+ 10*N &+ &D \\
&+ 10*R &+ &E \\
9000*M &+ 900*O &+ 100*N &+ 10*E &+ Y
\end{align*}
\]
Normalization

\[
\begin{align*}
1000S + & 100E + D \\
+ & 10R + E \\
9000M + & 900O + 90N + 10E + Y
\end{align*}
\]
Normalization

\[
\begin{align*}
1000S+ & \quad 100E+ & \quad D \\
+ & \quad 10R+ & \quad E \\
\hline
9000M+ & \quad 900O+ & \quad 90N+ & \quad 10E+ & \quad Y
\end{align*}
\]
Normalization

\[
\begin{align*}
1000*S + 91*E + D + 10*R \\
9000*M + 900*O + 90*N + Y
\end{align*}
\]
Simplified Equation

\[ 1000 \times S + 91 \times E + 10 \times R + D = 9000 \times M + 900 \times O + 90 \times N + Y \]
Propagation

\[ 1000 \times S^{1..9} + 91 \times E^{0..9} + 10 \times R^{0..9} + D^{0..9} = \\
9000 \times M^{1..9} + 900 \times O^{0..9} + 90 \times N^{0..9} + Y^{0..9} \]
Propagation

\[
\begin{align*}
1000 \times S^{1..9} & + 91 \times E^{0..9} + 10 \times R^{0..9} + D^{0..9} = 1000..9918 \\
9000 \times M^{1..9} & + 900 \times O^{0..9} + 90 \times N^{0..9} + Y^{0..9} = 9000..89919
\end{align*}
\]
Propagation

\[
1000 \times S^{1..9} + 91 \times E^{0..9} + 10 \times R^{0..9} + D^{0..9} = 9000..9918
\]

\[
9000 \times M^{1..9} + 900 \times O^{0..9} + 90 \times N^{0..9} + Y^{0..9} = 9000..9918
\]
Propagation

\[1000 \times S^{1..9} + 91 \times E^{0..9} + 10 \times R^{0..9} + D^{0..9} = 9000..9918\]

\[9000 \times M^{1..9} + 900 \times O^{0..9} + 90 \times N^{0..9} + Y^{0..9} = 9000..9918\]

Deduction:

\[M = 1, \ S = 9, \ O \in \{0..1\}\]
Propagation

\[1000 \times S^{1..9} + 91 \times E^{0..9} + 10 \times R^{0..9} + D^{0..9} = \]
\[9000..9918\]
\[9000 \times M^{1..9} + 900 \times O^{0..9} + 90 \times N^{0..9} + Y^{0..9} = \]
\[9000..9918\]

Deduction:

\[M = 1, S = 9, O \in \{0..1\}\]

Why? 

Skip
Consider lower bound for $S$

\[
\frac{1000 \times S^{1..9} + 91 \times E^{0..9} + 10 \times R^{0..9} + D^{0..9}}{9000..9918} = \frac{9000 \times M^{1..9} + 900 \times O^{0..9} + 90 \times N^{0..9} + Y^{0..9}}{9000..9918}
\]

- Lower bound of equation is 9000
- Rest of lhs (left hand side) \((91 \times E^{0..9} + 10 \times R^{0..9} + D^{0..9})\) is atmost 918
- \(S\) must be greater or equal to \(\frac{9000-918}{1000} = 8.082\)
  - otherwise lower bound of equation not reached by lhs
- \(S\) is integer, therefore \(S \geq \left\lceil \frac{9000-918}{1000} \right\rceil = 9\)
- \(S\) has upper bound of 9, so \(S = 9\)
Consider upper bound of $M$

\[
\begin{align*}
1000 \cdot S^{1..9} &+ 91 \cdot E^{0..9} + 10 \cdot R^{0..9} + D^{0..9} = 9000 \cdot M^{1..9} + 900 \cdot O^{0..9} + 90 \cdot N^{0..9} + Y^{0..9} \\
9000..9918 &\quad 9000..9918 
\end{align*}
\]

- Upper bound of equation is 9918
- Rest of rhs (right hand side) $900 \cdot O^{0..9} + 90 \cdot N^{0..9} + Y^{0..9}$ is at least 0
- $M$ must be smaller or equal to $\frac{9918-0}{9000} = 1.102$
- $M$ must be integer, therefore $M \leq \lfloor \frac{9918-0}{9000} \rfloor = 1$
- $M$ has lower bound of 1, so $M = 1$
Consider upper bound of $O$

$$\underbrace{1000 \times S^{1..9} + 91 \times E^{0..9} + 10 \times R^{0..9} + D^{0..9}}_{9000..9918} = \underbrace{9000 \times M^{1..9} + 900 \times O^{0..9} + 90 \times N^{0..9} + Y^{0..9}}_{9000..9918}$$

- Upper bound of equation is 9918
- Rest of rhs (right hand side) $9000 \times 1 + 90 \times N^{0..9} + Y^{0..9}$ is at least 9000
- $O$ must be smaller or equal to $\frac{9918 - 9000}{900} = 1.02$
- $O$ must be integer, therefore $O \leq \lfloor \frac{9918 - 9000}{900} \rfloor = 1$
- $O$ has lower bound of 0, so $O \in \{0..1\}$
## Propagation of equality: Result

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>S</strong></td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>🌟</td>
</tr>
<tr>
<td><strong>E</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>N</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>D</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>M</strong></td>
<td>🌟</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td><strong>O</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>R</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Y</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Helmut Simonis  | Basic Constraint Reasoning  | 60
### Propagation of alldifferent

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>S</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>E</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>N</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>D</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>M</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>O</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>R</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Y</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

- S
- E
- N
- D
- M
- O
- R
- Y

**S** - **E** - **N** - **M** - **O**

```
S
E
N
D
M
O
R
Y
```
Propagation of alldifferent

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>S</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>*</td>
</tr>
<tr>
<td>E</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>N</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>D</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>M</td>
<td></td>
<td></td>
<td>*</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>O</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>R</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Y</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Helmut Simonis
Basic Constraint Reasoning
### Propagation of alldifferent

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>S</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td><strong>R</strong></td>
</tr>
<tr>
<td><strong>E</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>N</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>D</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>M</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td><strong>Y</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>O</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>R</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Note: **M** is the starting point, and the propagation of the alldifferent constraint affects the values in the grid.
## Propagation of alldifferent

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>S</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>E</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>N</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>D</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>M</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>O</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>R</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Y</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The image shows a grid with letters and numbers, illustrating the propagation of the alldifferent constraint in a constraint satisfaction problem. The grid is partially filled with letters and numbers, indicating constraints that have been applied or inferred.
Propagation of alldifferent

<table>
<thead>
<tr>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>S</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>E</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>N</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>D</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>M</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>O</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>R</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Y</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Helmut Simonis
Basic Constraint Reasoning
Propagating the alldifferent constraint

O = 0, [E, R, D, N, Y] ∈ \{2..8\}
Waking the equality constraint

- Triggered by assignment of variables
- or update of lower or upper bound
Removal of constants

\[
1000 \times 9 + 91 \times E^{2..8} + 10 \times R^{2..8} + D^{2..8} = \\
9000 \times 1 + 900 \times 0 + 90 \times N^{2..8} + Y^{2..8}
\]
Removal of constants

\[1000 \times 9 + 91 \times E^{2..8} + 10 \times R^{2..8} + D^{2..8} =
9000 \times 1 + 900 \times 0 + 90 \times N^{2..8} + \gamma^{2..8}\]
Removal of constants

\[ 91 \times E^{2..8} + 10 \times R^{2..8} + D^{2..8} = 90 \times N^{2..8} + Y^{2..8} \]
Propagation of equality (Iteration 1)

\[
91 \times E^{2..8} + 10 \times R^{2..8} + D^{2..8} = 90 \times N^{2..8} + Y^{2..8}
\]

\[
\begin{array}{c}
204..816 \\
182..728
\end{array}
\]
Propagation of equality (Iteration 1)

\[
\begin{align*}
91 \times E^{2..8} + 10 \times R^{2..8} + D^{2..8} &= 90 \times N^{2..8} + Y^{2..8} \\
204..728 &
\end{align*}
\]
Propagation of equality (Iteration 1)

\[
91 \times E_{2..8} + 10 \times R_{2..8} + D_{2..8} = 90 \times N_{2..8} + Y_{2..8} \\
\text{204..728}
\]

\[
N \geq 3 = \left\lfloor \frac{204 - 8}{90} \right\rfloor, \quad E \leq 7 = \left\lfloor \frac{728 - 22}{91} \right\rfloor
\]
Propagation of equality (Iteration 2)

\[91 \times E^{2..7} + 10 \times R^{2..8} + D^{2..8} = 90 \times N^{3..8} + Y^{2..8}\]
Propagation of equality (Iteration 2)

\[ 91 \times E^{2..7} + 10 \times R^{2..8} + D^{2..8} = 90 \times N^{3..8} + Y^{2..8} \]
Propagation of equality (Iteration 2)

\[ 91 \times E^{2..7} + 10 \times R^{2..8} + D^{2..8} = 90 \times N^{3..8} + Y^{2..8} \]

\[ 272..725 \]
Propagation of equality (Iteration 2)

\[
91 \times E^{2..7} + 10 \times R^{2..8} + D^{2..8} = 90 \times N^{3..8} + Y^{2..8}
\]

\[
E \geq 3 = \left\lfloor \frac{272 - 88}{91} \right\rfloor
\]
Propagation of equality (Iteration 3)

\[ 91 \times E^{3..7} + 10 \times R^{2..8} + D^{2..8} = 90 \times N^{3..8} + Y^{2..8} \]
Propagating equality (Iteration 3)

\[
91 \times E^{3..7} + 10 \times R^{2..8} + D^{2..8} = 90 \times N^{3..8} + Y^{2..8}
\]
Propagation of equality (Iteration 3)

\[
\begin{align*}
91 \times E^{3..7} + 10 \times R^{2..8} + D^{2..8} &= 90 \times N^{3..8} + Y^{2..8} \\
295..725
\end{align*}
\]
Propagation of equality (Iteration 3)

\[
\begin{align*}
91 \times E^{3..7} + 10 \times R^{2..8} + D^{2..8} &= 90 \times N^{3..8} + Y^{2..8} \\
295..725 &= 90 \\
N \geq 4 &= \left\lceil \frac{295 - 8}{90} \right\rceil
\end{align*}
\]
Propagation of equality (Iteration 4)

\[ 91 \cdot E^{3..7} + 10 \cdot R^{2..8} + D^{2..8} = 90 \cdot N^{4..8} + Y^{2..8} \]
Propagation of equality (Iteration 4)

\[ 91 \times E^{3..7} + 10 \times R^{2..8} + D^{2..8} = 90 \times N^{4..8} + Y^{2..8} \]

\[ 295..725 = 362..728 \]
Propagation of equality (Iteration 4)

\[
91 \cdot E^3..7 + 10 \cdot R^2..8 + D^2..8 = 90 \cdot N^4..8 + Y^2..8
\]

\[
362..725
\]
Propagation of equality (Iteration 4)

\[
91 \times E^{3..7} + 10 \times R^{2..8} + D^{2..8} = 90 \times N^{4..8} + Y^{2..8} \\
\text{362..725}
\]

\[
E \geq 4 = \left\lceil \frac{362 - 88}{91} \right\rceil
\]
Propagation of equality (Iteration 5)

\[ 91 \times E^{4..7} + 10 \times R^{2..8} + D^{2..8} = 90 \times N^{4..8} + Y^{2..8} \]
Propagation of equality (Iteration 5)

\[
91 \times E^{4..7} + 10 \times R^{2..8} + D^{2..8} = 90 \times N^{4..8} + Y^{2..8}
\]

\[
\begin{align*}
386..725 \\
362..728
\end{align*}
\]
Propagation of equality (Iteration 5)

\[ 91 \times E^{4..7} + 10 \times R^{2..8} + D^{2..8} = 90 \times N^{4..8} + Y^{2..8} \]

or

\[ 386..725 \]
Propagation of equality (Iteration 5)

\[
91 \times E^{4\ldots 7} + 10 \times R^{2\ldots 8} + D^{2\ldots 8} = 90 \times N^{4\ldots 8} + Y^{2\ldots 8} \\
\underbrace{386..725}
\]

\[
N \geq 5 = \left\lfloor \frac{386 - 8}{90} \right\rfloor
\]
Propagation of equality (Iteration 6)

\[ 91 \times E^{4.7} + 10 \times R^{2.8} + D^{2.8} = 90 \times N^{5.8} + Y^{2.8} \]
Propagation of equality (Iteration 6)

\[ 91 \times E^{4..7} + 10 \times R^{2..8} + D^{2..8} = 90 \times N^{5..8} + Y^{2..8} \]

\[ \begin{align*}
386..725 \\
452..728
\end{align*} \]
Propagation of equality (Iteration 6)

\[ 91 \times E^{4..7} + 10 \times R^{2..8} + D^{2..8} = 90 \times N^{5..8} + Y^{2..8} \]

\[ 452..725 \]
Propagation of equality (Iteration 6)

\[
\begin{align*}
91 \cdot E^{4..7} + 10 \cdot R^{2..8} + D^{2..8} &= 90 \cdot N^{5..8} + Y^{2..8} \\
452..725
\end{align*}
\]

\[
N \geq 5 = \left\lceil \frac{452 - 8}{90} \right\rceil, \quad E \geq 4 = \left\lceil \frac{452 - 88}{91} \right\rceil
\]

No further propagation at this point
Domains after setup

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>S</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>E</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>N</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>D</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>M</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>O</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>R</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Y</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Outline

1. Problem
2. Program
3. Constraint Setup
4. Search
   - Step 1
   - Step 2
   - Further Steps
   - Solution
5. Lessons Learned
labeling built-in

labeling([S,E,N,D,M,O,R,Y])

- Try variable is order given
- Try values starting from smallest value in domain
- When failing, backtrack to last open choice
- Chronological Backtracking
- Depth First search
Search Tree Step 1

Variable $S$ already fixed
Step 2, Alternative $E = 4$

Variable $E \in \{4..7\}$, first value tested is 4
Assignment $E = 4$
Propagation of $E = 4$, equality constraint

$$91 \times 4 + 10 \times R^{2..8} + D^{2..8} = 90 \times N^{5..8} + Y^{2..8}$$
Propagation of $E = 4$, equality constraint

\[
\underbrace{91 \times 4 + 10 \times R^{2..8} + D^{2..8}}_{386..452} = \underbrace{90 \times N^{5..8} + Y^{2..8}}_{452..728}
\]
Propagation of $E = 4$, equality constraint

$$91 \times 4 + 10 \times R^{2..8} + D^{2..8} = 90 \times N^{5..8} + Y^{2..8}$$
Propagation of $E = 4$, equality constraint

\[
\begin{align*}
91 \times 4 + 10 \times R^{2..8} + D^{2..8} &= 90 \times N^{5..8} + Y^{2..8} \\
452 &= 90 \times N^{5..8} + Y^{2..8}
\end{align*}
\]

$N = 5, Y = 2, R = 8, D = 8$
### Result of equality propagation

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>S</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td><strong>R</strong></td>
</tr>
<tr>
<td>E</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td><strong>D</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>N</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td><strong>M</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>D</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td><strong>Y</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>M</td>
<td><strong>H</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>O</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td><strong>R</strong></td>
<td></td>
</tr>
<tr>
<td>R</td>
<td><strong>E</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Y</td>
<td></td>
<td><strong>S</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

*Note: The red stars indicate the result of the equality propagation.*
Propagation of alldifferent

```
   0 1 2 3 4 5 6 7 8 9
S    - - - - - - - -
E    - - - - - - - -
N    - - - - - - - -
D    - - - - - - - -
M    - - - - - - - -
O    - - - - - - - -
R    - - - - - - - -
Y    - - - - - - - -
```
## Propagation of \textit{alldifferent}

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>S</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>E</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>N</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>D</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>M</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>O</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>R</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Y</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Alldifferent fails!
Step 2, Alternative $E = 5$

Return to last open choice, $E$, and test next value
Assignment $E = 5$
## Propagation of alldifferent

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>S</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>E</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>N</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>D</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>M</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>-</td>
</tr>
<tr>
<td>O</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>-</td>
</tr>
<tr>
<td>R</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Y</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Helmut Simonis | Basic Constraint Reasoning
# Propagation of alldifferent

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>S</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>E</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>N</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>D</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>M</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>O</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>R</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Y</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Helmut Simonis

Basic Constraint Reasoning
Propagating the alldifferent constraint:

\[ N \neq 5, N \geq 6 \]
Propagation of equality

\[91 \times 5 + 10 \times R^{2..8} + D^{2..8} = 90 \times N^{6..8} + Y^{2..8}\]
Propagation of equality

\[
91 \times 5 + 10 \times R^{2..8} + D^{2..8} = 90 \times N^{6..8} + Y^{2..8}
\]

\[
\frac{477..543}{542..728}
\]
91 \times 5 + 10 \times R^{2..8} + D^{2..8} = 90 \times N^{6..8} + Y^{2..8} 
\hline
542..543
Propagation of equality

\[
91 \times 5 + 10 \times R^{2..8} + D^{2..8} = 90 \times N^{6..8} + Y^{2..8}
\]

\[
\text{542..543}
\]

\[
N = 6, \ Y \in \{2, 3\}, \ R = 8, \ D \in \{7..8\}
\]
Result of equality propagation

```
<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>S</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>E</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>N</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>D</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>M</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>O</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>R</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Y</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
```

- Stars represent equalities, and crosses represent constraints that have been propagated.

Helmut Simonis
Basic Constraint Reasoning
### Propagation of all different

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>S</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>E</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>N</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>*</td>
<td></td>
</tr>
<tr>
<td>D</td>
<td></td>
<td></td>
<td>*</td>
<td></td>
<td>*</td>
<td>*</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>M</td>
<td>*</td>
<td>*</td>
<td></td>
<td></td>
<td>*</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>O</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>R</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>*</td>
<td>*</td>
<td></td>
</tr>
<tr>
<td>Y</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>*</td>
<td>*</td>
<td>*</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Notes:
- The red squares indicate the propagated values.
- The white squares represent the remaining values.
- The black squares indicate the constraints that have been applied.
Propagation of \textit{alldifferent}
### Propagation of `alldifferent`

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>S</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>E</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>N</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>M</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>O</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>R</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Y</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

- **Step 1**
- **Step 2**
- **Further Steps**
- **Solution**

---

**Helmut Simonis**

**Basic Constraint Reasoning**
Propagation of \texttt{alldifferent}

\[ D = 7 \]
Propagation of equality

\[ 91 \times 5 + 10 \times 8 + 7 = 90 \times 6 + Y^{2,3} \]
Propagation of equality

\[ 91 \times 5 + 10 \times 8 + 7 = 90 \times 6 + Y^{2..3} \]

542

542..543
Propagation of equality

\[ 91 \times 5 + 10 \times 8 + 7 = 90 \times 6 + Y^{2..3} \]

542
Propagation of equality

\[ 91 \times 5 + 10 \times 8 + 7 = 90 \times 6 + Y^{2\ldots3} \]

\[ 542 \]

\[ Y = 2 \]
Last propagation step

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>S</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>E</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>N</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>D</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>M</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>O</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>R</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Y</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

S: SEND
E: END
N: END
D: END
M: END
O: END
R: END
Y: END

Solution
Further Steps: Nothing more to do
Further Steps: Nothing more to do
Further Steps: Nothing more to do
Further Steps: Nothing more to do
Further Steps: Nothing more to do
Further Steps: Nothing more to do
Further Steps: Nothing more to do
Complete Search Tree

The diagram represents a search tree with the following nodes and labels:

- S
- E
- N
- D
- M
- O
- R
- Y
- 2
- 3
- 4
- 5
- 6
- 7
- 8
- 9

The tree structure and the labels indicate the search process and decision points in a constraint reasoning problem.
Solution

\[
\begin{array}{cccccc}
9 & 5 & 6 & 7 & + & 1 & 0 & 8 & 5 \\
\hline
1 & 0 & 6 & 5 & 2
\end{array}
\]
Topics introduced

- Finite Domain Solver in ECLiPSe, \texttt{ic} library
- Models and Programs
- Constraint Propagation and Search
- Basic constraints: linear arithmetic, \texttt{alldifferent}, disequality
- Built-in search: \texttt{labeling}
- Visualizers for variables, constraints and search
Lessons Learned

- Constraint models are expressed by variables and constraints.
- Problems can have many different models, which can behave quite differently. Choosing the best model is an art.
- Constraints can take many different forms.
- Propagation deals with the interaction of variables and constraints.
- It removes some values that are inconsistent with a constraint from the domain of a variable.
- Constraints only communicate via shared variables.
Lessons Learned

- Propagation usually is not sufficient, search may be required to find a solution.
- Propagation is data driven, and can be quite complex even for small examples.
- The default search uses chronological depth-first backtracking, systematically exploring the complete search space.
- The search choices and propagation are interleaved, after every choice some more propagation may further reduce the problem.
Outline

6 Alternative Models
- Model without Disequality
- Multiple Equations

7 Exercises
Alternative 1

- Do we need the constraint “Numbers do not begin with a zero”?
- This is not given explicitly in the problem statement
- Remove disequality constraints from program
- Previous solution is still a solution
- Does it change propagation?
- Does it have more solutions?
Program without Disequality

Listing 1: Alternative 1

:- module(alternative1).
:- export(sendmory/1).
:- lib(ic).

sendmory(L):-
    L = [S,E,N,D,M,O,R,Y],
    L :: 0..9,
    alldifferent(L),
    1000*S + 100*E + 10*N + D +
    1000*M + 100*O + 10*R + E #=  
    10000*M + 1000*O + 100*N + 10*E + Y,
    labeling(L).
After Setup without Disequality

```
<table>
<thead>
<tr>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>S</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>E</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>N</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>D</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>M</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>O</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>R</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Y</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
```
Setup Comparison

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>S</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>E</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>N</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>D</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>M</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>O</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>R</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Y</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

original

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>S</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>E</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>N</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>D</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>M</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>O</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>R</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Y</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

alternative 1

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>S</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>E</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>N</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>D</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>M</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>O</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>R</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Y</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Search Tree: Many Solutions

S
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions

Helmut Simonis  Basic Constraint Reasoning
Search Tree: Many Solutions

0 1 2
0
1
2
Search Tree: Many Solutions
Search Tree: Many Solutions

\begin{align*}
S & \quad 0 \quad 1 \quad 2 \\
E & \quad 8 \\
N & \quad 1 \\
D & \quad 7 \\
M & \end{align*}
Search Tree: Many Solutions

[Diagram of a search tree with nodes labeled S, E, N, D, M, O at levels 0, 1, 2, 7, 8, 1, respectively.]

Back to Start  Skip Animation
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions

Alternative Models
Exercises

Model without Disequality
Multiple Equations

Helmut Simonis
Basic Constraint Reasoning 156
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions

Helmut Simonis
Basic Constraint Reasoning 164
Search Tree: Many Solutions

Diagram showing a search tree with many solutions.
Search Tree: Many Solutions

```
S
0 1 2 3
E E
8 7
N N
1 1
D D
7 9 2
M M M M
0 0 0
O O O
3 3 4
R R R
6 6 6
Y Y Y
5 7
```

Back to Start  Skip Animation
Search Tree: Many Solutions

Alternative Models
Exercises
Model without Disequality
Multiple Equations

Helmut Simonis
Basic Constraint Reasoning
Search Tree: Many Solutions

Helmut Simonis
Basic Constraint Reasoning
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions

Alternative Models
Exercises

Model without Disequality
Multiple Equations
Search Tree: Many Solutions
Search Tree: Many Solutions
Alternative Models
Exercises

Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions

- **Model without Disequality**
- **Multiple Equations**

- **Alternative Models**
- **Exercises**
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Alternative Models
Exercises

Model without Disequality
Multiple Equations

Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions

Back to Start  Skip Animation
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
search tree: many solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions

Alternative Models
Exercises

Model without Disequality
Multiple Equations
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Not just a different model, solving a different problem!
Often we can choose which problem we want to solve
- Which constraints to include
- What to ignore
In this case not acceptable
Alternative 2

- Large equality difficult to understand by humans
- Replace with multiple, simpler equations
- Linked by carry variables (0/1)
- Should produce same solutions
- Does it give same propagation?

\[
\begin{array}{c}
S & E & N & D \\
+ & M & O & R & E \\
+ & C_5 & C_4 & C_3 & C_2 \\
\hline
M & O & N & E & Y
\end{array}
\]
Carry Variables with Multiple Equations

```prolog
:-module(alternative2),export(sendmory/1),lib(ic).
sendmory(L):-
    L=[S,E,N,D,M,O,R,Y],L :: 0..9,
    [C2,C3,C4,C5] :: 0..1,
    alldifferent(L),
    S #\= 0,M #\= 0,
    M #= C5,
    S+M+C4 #= 10*C5+O,
    E+O+C3 #= 10*C4+N,
    N+R+C2 #= 10*C3+E,
    D+E #= 10*C2+Y,
    labeling(L).
```
Carry Variables with Multiple Equations

:-module(alternative2), export(sendmory/1), lib(ic).

sendmory(L):-
  same as before
  L=[S,E,N,D,M,O,R,Y], L :: 0..9,
  [C2,C3,C4,C5] :: 0..1,
  alldifferent(L),
  S #\= 0, M #\= 0,
  M #= C5,
  S+M+C4 #= 10*C5+O,
  E+O+C3 #= 10*C4+N,
  N+R+C2 #= 10*C3+E,
  D+E #= 10*C2+Y,
  labeling(L).
Carry Variables with Multiple Equations

:-module(alternative2),export(sendmory/1),lib(ic).

sendmory(L):-
    L=[S,E,N,D,M,O,R,Y],L :: 0..9,
    [C2,C3,C4,C5] :: 0..1, new
    alldifferent(L),
    S #\= 0,M #\= 0,
    M #= C5,
    S+M+C4 #= 10*C5+O,
    E+O+C3 #= 10*C4+N,
    N+R+C2 #= 10*C3+E,
    D+E #= 10*C2+Y,
    labeling(L).

  S  E  N  D
  +  M  O  R  E
  + C5  C4  C3  C2
  ---------------
  M  O  N  E  Y
:-module(alternative2),export(sendmory/1),lib(ic).

sendmory(L):-
L=[S,E,N,D,M,O,R,Y],L :: 0..9,
[C2,C3,C4,C5] :: 0..1,

call((equation/1))

call((equation2/1))

call((equation3/1))

call((equation4/1))

call((equation5/1))

labeling(L).
With Carry Variables: After Setup

```
  0 1 2 3 4 5 6 7 8 9
S  |
E  |
N  |
D  |
M  
O  |
R  |
Y  
```

Helmut Simonis

Basic Constraint Reasoning
Setup Comparison

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>S</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>E</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>N</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>D</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>M</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>O</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>R</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Y</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

original

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>S</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>E</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>N</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>D</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>M</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>O</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>R</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Y</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

alternative2
Search Tree: First Solution
Comparison

Single Equation

Multiple Equations
Observations

- This is solving the original problem
- Search tree slightly bigger
- Caused here by missing interaction of equations
- And repeated variables
- But: Introducing auxiliary variables not always bad!
Henry Dudeney. 
Send+More=Money. 
*Strand Magazine*, Volume 68:pages 97 and 214, July 1924.

Henry Dudeney. 
*Amusements in Mathematics.* 
Project Gutenberg, 1917. 
http://www.gutenberg.org/etext/16713.
Outline

6 Alternative Models

7 Exercises
Exercises

1. Does the reasoning for the equality constraints that we have presented remove all inconsistent values? Consider the constraint $Y=2\times X$.

2. Why is it important to remove multiple occurrences of the same variable from an equality constraint? Give an example!

3. Solve the puzzle DONALD + GERALD = ROBERT. What is the state of the variables before the search, after the initial constraint propagation?

4. Solve the puzzle $Y \times WORRY = DOOOOD$. What is different?

5. (extra credit) How would you design a program that finds new crypt-arithmetic puzzles? What makes a good puzzle?